

Autocommuting probability of a finite group relative to its subgroups

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Abstract

Let $H \subseteq K$ be two subgroups of a finite group G and $\text{Aut}(K)$ the automorphism group of K . The autocommuting probability of G relative to its subgroups H and K , denoted by $\text{Pr}(H, \text{Aut}(K))$, is the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\text{Aut}(K)$, is equal to the identity element of G . In this paper, we study $\text{Pr}(H, \text{Aut}(K))$ through a generalization.

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1 Introduction

Let G be a finite group acting on a set Ω . Let $\text{Pr}(G, \Omega)$ denote the probability that a randomly chosen element of Ω fixes a randomly chosen element of G . In 1975, Sherman [11] initiated the study of $\text{Pr}(G, \Omega)$ considering G to be an abelian group and $\Omega = \text{Aut}(G)$, the automorphism group of G . Note that

$$\text{Pr}(G, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \text{Aut}(G) : [x, \alpha] = 1\}|}{|G| |\text{Aut}(G)|}$$

where $[x, \alpha]$ is the autocommutator of x and α defined as $x^{-1}\alpha(x)$. The ratio $\text{Pr}(G, \text{Aut}(G))$ is called autocommuting probability of G .

Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Motivated by the works in [2, 6], we define

$$\text{Pr}_g(H, \text{Aut}(K)) = \frac{|\{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = g\}|}{|H| |\text{Aut}(K)|} \quad (1.1)$$

where $g \in K$. That is, $\text{Pr}_g(H, \text{Aut}(K))$ is the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\text{Aut}(K)$, is

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equal to a given element $g \in K$. The ratio $\text{Pr}_g(H, \text{Aut}(K))$ is called generalized autocommuting probability of G relative to its subgroups H and K . Clearly, if $H = G$ and $g = 1$ then $\text{Pr}_g(H, \text{Aut}(K)) = \text{Pr}(G, \text{Aut}(G))$. We would like to mention here that the case when $H = G$ is considered in [3]. In this paper, we study $\text{Pr}_g(H, \text{Aut}(K))$ extensively. In particular, we obtain some computing formulae, various bounds and a few characterizations of G through a subgroup. We conclude the paper describing an invariance property of $\text{Pr}_g(H, \text{Aut}(K))$.

We write $S(H, \text{Aut}(K))$ to denote the set $\{[x, \alpha] : x \in H \text{ and } \alpha \in \text{Aut}(K)\}$ and $[H, \text{Aut}(K)] := \langle S(H, \text{Aut}(K)) \rangle$. We also write $L(H, \text{Aut}(K)) := \{x \in H : [x, \alpha] = 1 \text{ for all } \alpha \in \text{Aut}(K)\}$ and $L(G) := L(G, \text{Aut}(G))$, the absolute center of G (see [5]). Note that $L(H, \text{Aut}(K))$ is a normal subgroup of H contained in $H \cap Z(K)$. Further, $L(H, \text{Aut}(K)) = \bigcap_{\alpha \in \text{Aut}(K)} C_H(\alpha)$, where $C_H(\alpha) = \{x \in H : [x, \alpha] = 1\}$ is a subgroup of H . Let $C_{\text{Aut}(K)}(x) := \{\alpha \in \text{Aut}(K) : \alpha(x) = x\}$ for $x \in H$ and $C_{\text{Aut}(K)}(H) = \{\alpha \in \text{Aut}(K) : \alpha(x) = x \text{ for all } x \in H\}$. Then $C_{\text{Aut}(K)}(x)$ is a subgroup of $\text{Aut}(K)$ and $C_{\text{Aut}(K)}(H) = \bigcap_{x \in H} C_{\text{Aut}(K)}(x)$.

Clearly, $\text{Pr}_g(H, \text{Aut}(K)) = 1$ if and only if $[H, \text{Aut}(K)] = \{1\}$ and $g = 1$ if and only if $H = L(H, \text{Aut}(K))$ and $g = 1$. Also, $\text{Pr}_g(H, \text{Aut}(K)) = 0$ if and only if $g \notin S(H, \text{Aut}(K))$. Therefore, we consider $H \neq L(H, \text{Aut}(K))$ and $g \in S(H, \text{Aut}(K))$ throughout the paper.

2 Some computing formulae

For any $x \in H$, let us define the set $T_{x,g}(H, K) = \{\alpha \in \text{Aut}(K) : [x, \alpha] = g\}$, where g is a fixed element of K . Note that $T_{x,1}(H, K) = C_{\text{Aut}(K)}(x)$. The following two lemmas play a crucial role in obtaining computing formula for $\text{Pr}_g(H, \text{Aut}(K))$.

Lemma 2.1. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $T_{x,g}(H, K) \neq \emptyset$ then $T_{x,g}(H, K) = \sigma C_{\text{Aut}(K)}(x)$ for some $\sigma \in T_{x,g}(H, K)$ and hence $|T_{x,g}(H, K)| = |C_{\text{Aut}(K)}(x)|$.*

Proof. Let $\sigma \in T_{x,g}(H, K)$ and $\beta \in \sigma C_{\text{Aut}(K)}(x)$. Then $\beta = \sigma\alpha$ for some $\alpha \in C_{\text{Aut}(K)}(x)$. We have

$$[x, \beta] = [x, \sigma\alpha] = x^{-1}\sigma(\alpha(x)) = [x, \sigma] = g.$$

Therefore, $\beta \in T_{x,g}(H, K)$ and so $\sigma C_{\text{Aut}(K)}(x) \subseteq T_{x,g}(H, K)$. Again, let $\gamma \in T_{x,g}(H, K)$ then $\gamma(x) = xg$. We have $\sigma^{-1}\gamma(x) = \sigma^{-1}(xg) = x$ and so $\sigma^{-1}\gamma \in C_{\text{Aut}(K)}(x)$. Therefore, $\gamma \in \sigma C_{\text{Aut}(K)}(x)$ which gives $T_{x,g}(H, K) \subseteq \sigma C_{\text{Aut}(K)}(x)$. Hence, the result follows. \square

Consider the action of $\text{Aut}(K)$ on K given by $(\alpha, x) \mapsto \alpha(x)$ where $\alpha \in \text{Aut}(K)$ and $x \in K$. Let $\text{orb}_K(x) := \{\alpha(x) : \alpha \in \text{Aut}(K)\}$ be the orbit of $x \in K$. Then by orbit-stabilizer theorem, we have

$$|\text{orb}_K(x)| = \frac{|\text{Aut}(K)|}{|C_{\text{Aut}(K)}(x)|}. \quad (2.1)$$

Lemma 2.2. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then $T_{x,g}(H, K) \neq \emptyset$ if and only if $xg \in \text{orb}_K(x)$.*

Proof. The result follows from the fact that $\alpha \in T_{x,g}(H, K)$ if and only if $xg \in \text{orb}_K(x)$. \square

The following theorem gives two computing formulae for $\text{Pr}_g(H, \text{Aut}(K))$.

Theorem 2.3. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\begin{aligned} \text{Pr}_g(H, \text{Aut}(K)) &= \frac{1}{|H| |\text{Aut}(K)|} \sum_{\substack{x \in H \\ xg \in \text{orb}_K(x)}} |C_{\text{Aut}(K)}(x)| \\ &= \frac{1}{|H|} \sum_{\substack{x \in H \\ xg \in \text{orb}_K(x)}} \frac{1}{|\text{orb}_K(x)|}. \end{aligned}$$

Proof. We have $\{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = g\} = \bigsqcup_{x \in H} (\{x\} \times T_{x,g}(H, K))$, where \sqcup represents the union of disjoint sets. Therefore, by (1.1), we have

$$|H| |\text{Aut}(K)| \text{Pr}_g(H, \text{Aut}(K)) = |\bigsqcup_{x \in H} (\{x\} \times T_{x,g}(H, K))| = \sum_{x \in H} |T_{x,g}(H, K)|.$$

Hence, the result follows from Lemma 2.1, Lemma 2.2 and (2.1). \square

Considering $g = 1$ in Theorem 2.3, we get the following computing formulae for $\text{Pr}(H, \text{Aut}(K))$.

Corollary 2.4. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\text{Pr}(H, \text{Aut}(K)) = \frac{1}{|H| |\text{Aut}(K)|} \sum_{x \in H} |C_{\text{Aut}(K)}(x)| = \frac{|\text{orb}_K(H)|}{|H|}$$

where $\text{orb}_K(H) = \{\text{orb}_K(x) : x \in H\}$.

Corollary 2.5. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $C_{\text{Aut}(K)}(x) = \{I\}$ for all $x \in H \setminus \{1\}$, where I is the identity element of $\text{Aut}(K)$, then*

$$\text{Pr}(H, \text{Aut}(K)) = \frac{1}{|H|} + \frac{1}{|\text{Aut}(K)|} - \frac{1}{|H| |\text{Aut}(K)|}.$$

Proof. By Corollary 2.4, we have

$$|H| |\text{Aut}(K)| \text{Pr}(H, \text{Aut}(K)) = \sum_{x \in H} |C_{\text{Aut}(K)}(x)| = |\text{Aut}(K)| + |H| - 1.$$

Hence, the result follows. \square

We also have $|\{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = 1\}| = \sum_{\alpha \in \text{Aut}(K)} |C_H(\alpha)|$ and hence

$$\Pr(H, \text{Aut}(K)) = \frac{1}{|H| |\text{Aut}(K)|} \sum_{\alpha \in \text{Aut}(K)} |C_H(\alpha)|. \quad (2.2)$$

We conclude this section with the following two results.

Proposition 2.6. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\Pr_{g^{-1}}(H, \text{Aut}(K)) = \Pr_g(H, \text{Aut}(K)).$$

Proof. Let

$$\begin{aligned} A &= \{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = g\} \text{ and} \\ B &= \{(y, \beta) \in H \times \text{Aut}(K) : [y, \beta] = g^{-1}\}. \end{aligned}$$

Then $(x, \alpha) \mapsto (\alpha(x), \alpha^{-1})$ gives a bijection between A and B . Therefore $|A| = |B|$. Hence, the result follows from (1.1). \square

Proposition 2.7. *Let G_1 and G_2 be two finite groups. Let H_1, K_1 and H_2, K_2 be subgroups of G_1 and G_2 respectively such that $H_1 \subseteq K_1$, $H_2 \subseteq K_2$ and $\gcd(|K_1|, |K_2|) = 1$. If $(g_1, g_2) \in K_1 \times K_2$ then*

$$\Pr_{(g_1, g_2)}(H_1 \times H_2, \text{Aut}(K_1 \times K_2)) = \Pr_{g_1}(H_1, \text{Aut}(K_1)) \Pr_{g_2}(H_2, \text{Aut}(K_2)).$$

Proof. Let

$$\begin{aligned} \mathcal{X} &= \{((x, y), \alpha_{K_1 \times K_2}) \in (H_1 \times H_2) \times \text{Aut}(K_1 \times K_2) : \\ &\quad [(x, y), \alpha_{K_1 \times K_2}] = (g_1, g_2)\}, \\ \mathcal{Y} &= \{(x, \alpha_{K_1}) \in H_1 \times \text{Aut}(K_1) : [x, \alpha_{K_1}] = g_1\} \text{ and} \\ \mathcal{Z} &= \{(y, \alpha_{K_2}) \in H_2 \times \text{Aut}(K_2) : [y, \alpha_{K_2}] = g_2\}. \end{aligned}$$

Since $\gcd(|K_1|, |K_2|) = 1$, by Lemma 2.1 of [1], we have $\text{Aut}(K_1 \times K_2) = \text{Aut}(K_1) \times \text{Aut}(K_2)$. Therefore, for every $\alpha_{K_1 \times K_2} \in \text{Aut}(K_1 \times K_2)$ there exist unique $\alpha_{K_1} \in \text{Aut}(K_1)$ and $\alpha_{K_2} \in \text{Aut}(K_2)$ such that $\alpha_{K_1 \times K_2} = \alpha_{K_1} \times \alpha_{K_2}$, where $\alpha_{K_1} \times \alpha_{K_2}((x, y)) = (\alpha_{K_1}(x), \alpha_{K_2}(y))$ for all $(x, y) \in H_1 \times H_2$. Also, for all $(x, y) \in H_1 \times H_2$, we have $[(x, y), \alpha_{K_1 \times K_2}] = (g_1, g_2)$ if and only if $[x, \alpha_{K_1}] = g_1$ and $[y, \alpha_{K_2}] = g_2$. These leads to show that $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$. Therefore

$$\frac{|\mathcal{X}|}{|H_1 \times H_2| |\text{Aut}(K_1 \times K_2)|} = \frac{|\mathcal{Y}|}{|H_1| |\text{Aut}(K_1)|} \cdot \frac{|\mathcal{Z}|}{|H_2| |\text{Aut}(K_2)|}.$$

Hence, the result follows from (1.1). \square

3 Various bounds

In this section, we obtain various bounds for $\text{Pr}_g(H, \text{Aut}(K))$. We begin with the following lower bounds.

Proposition 3.1. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then, for $g \in K$, we have*

$$\begin{aligned} \text{(a)} \quad \text{Pr}_g(H, \text{Aut}(K)) &\geq \frac{|L(H, \text{Aut}(K))|}{|H|} + \frac{|C_{\text{Aut}(K)}(H)|(|H| - |L(H, \text{Aut}(K))|)}{|H||\text{Aut}(K)|} \text{ if } g = 1. \\ \text{(b)} \quad \text{Pr}_g(H, \text{Aut}(K)) &\geq \frac{|L(H, \text{Aut}(K))||C_{\text{Aut}(K)}(H)|}{|H||\text{Aut}(K)|} \text{ if } g \neq 1. \end{aligned}$$

Proof. Let \mathcal{C} denote the set $\{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = g\}$.

(a) We have $(L(H, \text{Aut}(K)) \times \text{Aut}(K)) \cup (H \times C_{\text{Aut}(K)}(H))$ is a subset of \mathcal{C} and $|(L(H, \text{Aut}(K)) \times \text{Aut}(K)) \cup (H \times C_{\text{Aut}(K)}(H))| = |L(H, \text{Aut}(K))||\text{Aut}(K)| + |C_{\text{Aut}(K)}(H)||H| - |L(H, \text{Aut}(K))||C_{\text{Aut}(K)}(H)|$. Hence, the result follows from (1.1).

(b) Since $g \in S(H, \text{Aut}(K))$ we have \mathcal{C} is non-empty. Let $(y, \beta) \in \mathcal{C}$ then $(y, \beta) \notin L(H, \text{Aut}(K)) \times C_{\text{Aut}(K)}(H)$ otherwise $[y, \beta] = 1$. It is easy to see that the coset $(y, \beta)(L(H, \text{Aut}(K)) \times C_{\text{Aut}(K)}(H))$ is a subset of \mathcal{C} having order $|L(H, \text{Aut}(K))||C_{\text{Aut}(K)}(H)|$. Hence, the result follows from (1.1). \square

Proposition 3.2. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\text{Pr}_g(H, \text{Aut}(K)) \leq \text{Pr}(H, \text{Aut}(K)).$$

The equality holds if and only if $g = 1$.

Proof. By Theorem 2.3, we have

$$\begin{aligned} \text{Pr}_g(H, \text{Aut}(K)) &= \frac{1}{|H||\text{Aut}(K)|} \sum_{\substack{x \in H \\ xg \in \text{orb}_K(x)}} |C_{\text{Aut}(K)}(x)| \\ &\leq \frac{1}{|H||\text{Aut}(K)|} \sum_{x \in H} |C_{\text{Aut}(K)}(x)| = \text{Pr}(H, \text{Aut}(K)). \end{aligned}$$

The equality holds if and only if $xg \in \text{orb}_K(x)$ for all $x \in H$ if and only if $g = 1$. \square

Proposition 3.3. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Let $g \in K$ and p the smallest prime dividing $|\text{Aut}(K)|$. If $g \neq 1$ then*

$$\text{Pr}_g(H, \text{Aut}(K)) \leq \frac{|H| - |L(H, \text{Aut}(K))|}{p|H|} < \frac{1}{p}.$$

Proof. By Theorem 2.3, we have

$$\Pr_g(H, \text{Aut}(K)) = \frac{1}{|H|} \sum_{\substack{x \in H \setminus L(H, \text{Aut}(K)) \\ xg \in \text{orb}_K(x)}} \frac{1}{|\text{orb}_K(x)|} \quad (3.1)$$

noting that for $x \in L(H, \text{Aut}(K))$ we have $xg \notin \text{orb}_K(x)$. Also, for $x \in H \setminus L(H, \text{Aut}(K))$ and $xg \in \text{orb}_K(x)$ we have $|\text{orb}_K(x)| > 1$. Since $|\text{orb}_K(x)|$ is a divisor of $|\text{Aut}(K)|$ we have $|\text{orb}_K(x)| \geq p$. Hence, the result follows from (3.1). \square

Proposition 3.4. *Let H_1 , H_2 and K be subgroups of a finite group G such that $H_1 \subseteq H_2 \subseteq K$. Then*

$$\Pr_g(H_1, \text{Aut}(K)) \leq |H_2 : H_1| \Pr_g(H_2, \text{Aut}(K)).$$

The equality holds if and only if $xg \notin \text{orb}_K(x)$ for all $x \in H_2 \setminus H_1$.

Proof. By Theorem 2.3, we have

$$\begin{aligned} |H_1| |\text{Aut}(K)| \Pr_g(H_1, \text{Aut}(K)) &= \sum_{\substack{x \in H_1 \\ xg \in \text{orb}_K(x)}} |C_{\text{Aut}(K)}(x)| \\ &\leq \sum_{\substack{x \in H_2 \\ xg \in \text{orb}_K(x)}} |C_{\text{Aut}(K)}(x)| \\ &= |H_2| |\text{Aut}(K)| \Pr_g(H_2, \text{Aut}(K)). \end{aligned}$$

Hence, the result follows. \square

Proposition 3.5. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\Pr_g(H, \text{Aut}(K)) \leq |K : H| \Pr(K, \text{Aut}(K))$$

with equality if and only if $g = 1$ and $H = K$.

Proof. By Proposition 3.2, we have

$$\begin{aligned} \Pr_g(H, \text{Aut}(K)) &\leq \Pr(H, \text{Aut}(K)) \\ &= \frac{1}{|H| |\text{Aut}(K)|} \sum_{x \in H} |C_{\text{Aut}(K)}(x)| \\ &\leq \frac{1}{|H| |\text{Aut}(K)|} \sum_{x \in K} |C_{\text{Aut}(K)}(x)|. \end{aligned}$$

Hence, the result follows from Corollary 2.4. \square

Note that if we replace $\text{Aut}(K)$ by $\text{Inn}(K)$, the inner automorphism group of K , in (1.1) then $\text{Pr}_g(H, \text{Inn}(K)) = \text{Pr}_g(H, K)$ where

$$\text{Pr}_g(H, K) = \frac{|\{(x, y) \in H \times K : x^{-1}y^{-1}xy = g\}|}{|H||K|}.$$

A detailed study on $\text{Pr}_g(H, K)$ can be found in [2]. The following proposition gives a relation between $\text{Pr}_g(H, \text{Aut}(K))$ and $\text{Pr}_g(H, K)$ for $g = 1$.

Proposition 3.6. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g = 1$ then*

$$\text{Pr}_g(H, \text{Aut}(K)) \leq \text{Pr}_g(H, K).$$

Proof. If $g = 1$ then by [2, Theorem 2.3], we have

$$\text{Pr}_g(H, K) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|\text{cl}_K(x)|} \quad (3.2)$$

where $\text{cl}_K(x) = \{\alpha(x) : \alpha \in \text{Inn}(K)\}$. Since $\text{cl}_K(x) \subseteq \text{orb}_K(x)$ for all $x \in H$, the result follows from (3.2) and Theorem 2.3. \square

Theorem 3.7. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$ and p the smallest prime dividing $|\text{Aut}(K)|$. Then*

$$\text{Pr}(H, \text{Aut}(K)) \geq \frac{|L(H, \text{Aut}(K))|}{|H|} + \frac{p(|H| - |X_H| - |L(H, \text{Aut}(K))|) + |X_H|}{|H||\text{Aut}(K)|}$$

and

$$\text{Pr}(H, \text{Aut}(K)) \leq \frac{(p-1)|L(H, \text{Aut}(K))| + |H|}{p|H|} - \frac{|X_H|(|\text{Aut}(K)| - p)}{p|H||\text{Aut}(K)|},$$

where $X_H = \{x \in H : C_{\text{Aut}(K)}(x) = \{I\}\}$.

Proof. We have $X_H \cap L(H, \text{Aut}(K)) = \emptyset$. Therefore

$$\begin{aligned} \sum_{x \in H} |C_{\text{Aut}(K)}(x)| &= |X_H| + |\text{Aut}(K)||L(H, \text{Aut}(K))| \\ &\quad + \sum_{x \in H \setminus (X_H \cup L(H, \text{Aut}(K)))} |C_{\text{Aut}(K)}(x)|. \end{aligned}$$

For $x \in H \setminus (X_H \cup L(H, \text{Aut}(K)))$ we have $\{I\} \neq C_{\text{Aut}(K)}(x) \neq \text{Aut}(K)$ which implies $p \leq |C_{\text{Aut}(K)}(x)| \leq \frac{|\text{Aut}(K)|}{p}$. Therefore

$$\begin{aligned} \sum_{x \in H} |C_{\text{Aut}(K)}(x)| &\geq |X_H| + |\text{Aut}(K)||L(H, \text{Aut}(K))| \\ &\quad + p(|H| - |X_H| - |L(H, \text{Aut}(K))|) \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \sum_{x \in H} |C_{\text{Aut}(K)}(x)| &\leq |X_H| + |\text{Aut}(K)| |L(H, \text{Aut}(K))| \\ &\quad + \frac{|\text{Aut}(K)| (|H| - |X_H| - |L(H, \text{Aut}(K))|)}{p}. \end{aligned} \quad (3.4)$$

Hence, the result follows from Corollary 2.4, (3.3) and (3.4). \square

We have the following two corollaries.

Corollary 3.8. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If p and q are the smallest primes dividing $|\text{Aut}(K)|$ and $|H|$ respectively then*

$$\Pr(H, \text{Aut}(K)) \leq \frac{p+q-1}{pq}.$$

In particular, if $p = q$ then $\Pr(H, \text{Aut}(K)) \leq \frac{2p-1}{p^2} \leq \frac{3}{4}$.

Proof. Since $H \neq L(H, \text{Aut}(K))$ we have $|H : L(H, \text{Aut}(K))| \geq q$. Therefore, by Theorem 3.7, we have

$$\Pr(H, \text{Aut}(K)) \leq \frac{1}{p} \left(\frac{p-1}{|H : L(H, \text{Aut}(K))|} + 1 \right) \leq \frac{p+q-1}{pq}.$$

\square

Corollary 3.9. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$ and p, q be the smallest primes dividing $|\text{Aut}(K)|$ and $|H|$ respectively. If H is non-abelian then*

$$\Pr(H, \text{Aut}(K)) \leq \frac{q^2 + p - 1}{pq^2}.$$

In particular, if $p = q$ then $\Pr(H, \text{Aut}(K)) \leq \frac{p^2+p-1}{p^3} \leq \frac{5}{8}$.

Proof. Since H is non-abelian we have $|H : L(H, \text{Aut}(K))| \geq q^2$. Therefore, by Theorem 3.7, we have

$$\Pr(H, \text{Aut}(K)) \leq \frac{1}{p} \left(\frac{p-1}{|H : L(H, \text{Aut}(K))|} + 1 \right) \leq \frac{q^2 + p - 1}{pq^2}.$$

\square

Now we obtain two lower bounds analogous to the lower bounds obtained in [9, Theorem A] and [8, Theorem 1].

Theorem 3.10. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\Pr(H, \text{Aut}(K)) \geq \frac{1}{|S(H, \text{Aut}(K))|} \left(1 + \frac{|S(H, \text{Aut}(K))| - 1}{|H : L(H, \text{Aut}(K))|} \right).$$

The equality holds if and only if $\text{orb}_K(x) = xS(H, \text{Aut}(K))$ for all $x \in H \setminus L(H, \text{Aut}(K))$.

Proof. For all $x \in H \setminus L(H, \text{Aut}(K))$ we have $\alpha(x) = x[x, \alpha] \in xS(H, \text{Aut}(K))$. Therefore $\text{orb}_K(x) \subseteq xS(H, \text{Aut}(K))$ and so $|\text{orb}_K(x)| \leq |S(H, \text{Aut}(K))|$ for all $x \in H \setminus L(H, \text{Aut}(K))$. Now, by Corollary 2.4, we have

$$\begin{aligned} \text{Pr}(H, \text{Aut}(K)) &= \frac{1}{|H|} \left(\sum_{x \in L(H, \text{Aut}(K))} \frac{1}{|\text{orb}_K(x)|} + \sum_{x \in H \setminus L(H, \text{Aut}(K))} \frac{1}{|\text{orb}_K(x)|} \right) \\ &\geq \frac{|L(H, \text{Aut}(K))|}{|H|} + \frac{1}{|H|} \sum_{x \in H \setminus L(H, \text{Aut}(K))} \frac{1}{|S(H, \text{Aut}(K))|}. \end{aligned}$$

Hence, the result follows. \square

Lemma 3.11. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then, for any two integers $m \geq n$, we have*

$$\frac{1}{n} \left(1 + \frac{n-1}{|H : L(H, \text{Aut}(K))|} \right) \geq \frac{1}{m} \left(1 + \frac{m-1}{|H : L(H, \text{Aut}(K))|} \right).$$

If $L(H, \text{Aut}(K)) \neq H$ then equality holds if and only if $m = n$.

Proof. The proof is an easy exercise. \square

Corollary 3.12. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\text{Pr}(H, \text{Aut}(K)) \geq \frac{1}{|[H, \text{Aut}(K)]|} \left(1 + \frac{|[H, \text{Aut}(K)]| - 1}{|H : L(H, \text{Aut}(K))|} \right).$$

If $H \neq L(H, \text{Aut}(K))$ then the equality holds if and only if $[H, \text{Aut}(K)] = S(H, \text{Aut}(K))$ and $\text{orb}_K(x) = x[H, \text{Aut}(K)]$ for all $x \in H \setminus L(H, \text{Aut}(K))$.

Proof. Since $|[H, \text{Aut}(K)]| \geq |S(H, \text{Aut}(K))|$, the result follows from Theorem 3.10 and Lemma 3.11.

Note that the equality holds if and only if equality holds in Theorem 3.10 and Lemma 3.11. \square

It is worth mentioning that Theorem 3.10 gives better lower bound than the lower bound given by Corollary 3.12. Also

$$\begin{aligned} \frac{1}{|[H, \text{Aut}(K)]|} \left(1 + \frac{|[H, \text{Aut}(K)]| - 1}{|H : L(H, \text{Aut}(K))|} \right) &\geq \frac{|L(H, \text{Aut}(K))|}{|H|} \\ &\quad + \frac{p(|H| - |L(H, \text{Aut}(K))|)}{|H| |\text{Aut}(K)|}. \end{aligned}$$

Hence, Theorem 3.10 gives better lower bound than the lower bound given by Theorem 3.7.

4 A few Characterizations

In this section, we obtain some characterizations of a subgroup H of G if equality holds in Corollary 3.8 and Corollary 3.9. We begin with the following result.

Theorem 4.1. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $\Pr(H, \text{Aut}(K)) = \frac{p+q-1}{pq}$ for some primes p and q . Then pq divides $|H||\text{Aut}(K)|$. Further, if p and q are the smallest primes dividing $|\text{Aut}(K)|$ and $|H|$ respectively, then*

$$\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q.$$

In particular, if H and $\text{Aut}(K)$ are of even order and $\Pr(H, \text{Aut}(K)) = \frac{3}{4}$ then $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_2$.

Proof. By (1.1), we have $(p+q-1)|H||\text{Aut}(K)| = pq|\{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = 1\}|$. Therefore, pq divides $|H||\text{Aut}(K)|$.

If p and q are the smallest primes dividing $|\text{Aut}(K)|$ and $|H|$ respectively then, by Theorem 3.7, we have

$$\frac{p+q-1}{pq} \leq \frac{1}{p} \left(\frac{p-1}{|H : L(H, \text{Aut}(K))|} + 1 \right)$$

which gives $|H : L(H, \text{Aut}(K))| \leq q$. Hence, $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q$. \square

Theorem 4.2. *Let $H \subseteq K$ be two subgroups of a finite group G such that H is non-abelian and $\Pr(H, \text{Aut}(K)) = \frac{q^2+p-1}{pq^2}$ for some primes p and q . Then pq divides $|H||\text{Aut}(K)|$. Further, if p and q are the smallest primes dividing $|\text{Aut}(K)|$ and $|H|$ respectively then*

$$\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q.$$

In particular, if H and $\text{Aut}(K)$ are of even order and $\Pr(H, \text{Aut}(K)) = \frac{5}{8}$ then $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. By (1.1), we have $(q^2+p-1)|H||\text{Aut}(K)| = pq^2|\{(x, \alpha) \in H \times \text{Aut}(K) : [x, \alpha] = 1\}|$. Therefore, pq divides $|H||\text{Aut}(K)|$.

If p and q are the smallest primes dividing $|\text{Aut}(K)|$ and $|H|$ respectively then, by Theorem 3.7, we have

$$\frac{q^2+p-1}{pq^2} \leq \frac{1}{p} \left(\frac{p-1}{|H : L(H, \text{Aut}(K))|} + 1 \right)$$

which gives $|H : L(H, \text{Aut}(K))| \leq q^2$. Since H is non-abelian we have $|H : L(H, \text{Aut}(K))| \neq 1, q$. Hence, $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. \square

The following two results give partial converses of Theorem 4.1 and 4.2 respectively.

Proposition 4.3. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Let p, q be the smallest prime divisors of $|\text{Aut}(K)|$, $|H|$ respectively and $|\text{Aut}(K) : C_{\text{Aut}(K)}(x)| = p$ for all $x \in H \setminus L(H, \text{Aut}(K))$.*

(a) *If $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q$ then $\text{Pr}(H, \text{Aut}(K)) = \frac{p+q-1}{pq}$.*

(b) *If $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$ then $\text{Pr}(H, \text{Aut}(K)) = \frac{q^2+p-1}{pq^2}$.*

Proof. Since $|\text{Aut}(K) : C_{\text{Aut}(K)}(x)| = p$ for all $x \in H \setminus L(H, \text{Aut}(K))$ we have $|C_{\text{Aut}(K)}(x)| = \frac{|\text{Aut}(K)|}{p}$ for all $x \in H \setminus L(H, \text{Aut}(K))$. Therefore, by Corollary 2.4, we have

$$\begin{aligned} \text{Pr}(H, \text{Aut}(K)) &= \frac{|L(H, \text{Aut}(K))|}{|H|} + \frac{1}{|H||\text{Aut}(K)|} \sum_{x \in H \setminus L(H, \text{Aut}(K))} |C_{\text{Aut}(K)}(x)| \\ &= \frac{|L(H, \text{Aut}(K))|}{|H|} + \frac{|H| - |L(H, \text{Aut}(K))|}{p|H|}. \end{aligned}$$

Thus

$$\text{Pr}(H, \text{Aut}(K)) = \frac{1}{p} \left(\frac{p-1}{|H : L(H, \text{Aut}(K))|} + 1 \right). \quad (4.1)$$

(a) If $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q$ then (4.1) gives $\text{Pr}(H, \text{Aut}(K)) = \frac{p+q-1}{pq}$.

(b) If $\frac{H}{L(H, \text{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$ then (4.1) gives $\text{Pr}(H, \text{Aut}(K)) = \frac{q^2+p-1}{pq^2}$. \square

5 Autoisoclinic pairs

In the year 1940, Hall [4] introduced the concept of isoclinism between two groups. Following Hall, Moghaddam et al. [7] have defined autoisoclinism between two groups, in the year 2013. Recall that two groups G_1 and G_2 are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{G_1}{L(G_1)} \rightarrow \frac{G_2}{L(G_2)}$, $\beta : [G_1, \text{Aut}(G_1)] \rightarrow [G_2, \text{Aut}(G_2)]$ and $\gamma : \text{Aut}(G_1) \rightarrow \text{Aut}(G_2)$ such that the following diagram commutes

$$\begin{array}{ccc} \frac{G_1}{L(G_1)} \times \text{Aut}(G_1) & \xrightarrow{\psi \times \gamma} & \frac{G_2}{L(G_2)} \times \text{Aut}(G_2) \\ \downarrow a_{(G_1, \text{Aut}(G_1))} & & \downarrow a_{(G_2, \text{Aut}(G_2))} \\ [G_1, \text{Aut}(G_1)] & \xrightarrow{\beta} & [G_2, \text{Aut}(G_2)] \end{array}$$

where the maps $a_{(G_i, \text{Aut}(G_i))} : \frac{G_i}{L(G_i)} \times \text{Aut}(G_i) \rightarrow [G_i, \text{Aut}(G_i)]$, for $i = 1, 2$, are given by

$$a_{(G_i, \text{Aut}(G_i))}(x_i L(G_i), \alpha_i) = [x_i, \alpha_i].$$

Such a pair $(\psi \times \gamma, \beta)$ is called an autoisoclinism between the groups G_1 and G_2 . We generalize the notion of autoisoclinism in the following way:

Let H_1, K_1 and H_2, K_2 be subgroups of the groups G_1 and G_2 respectively. The pairs of subgroups (H_1, K_1) and (H_2, K_2) such that $H_1 \subseteq K_1$ and $H_2 \subseteq K_2$ are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{H_1}{L(H_1, \text{Aut}(K_1))} \rightarrow \frac{H_2}{L(H_2, \text{Aut}(K_2))}$, $\beta : [H_1, \text{Aut}(K_1)] \rightarrow [H_2, \text{Aut}(K_2)]$ and $\gamma : \text{Aut}(K_1) \rightarrow \text{Aut}(K_2)$ such that the following diagram commutes

$$\begin{array}{ccc} \frac{H_1}{L(H_1, \text{Aut}(K_1))} \times \text{Aut}(K_1) & \xrightarrow{\psi \times \gamma} & \frac{H_2}{L(H_2, \text{Aut}(K_2))} \times \text{Aut}(K_2) \\ \downarrow a_{(H_1, \text{Aut}(K_1))} & & \downarrow a_{(H_2, \text{Aut}(K_2))} \\ [H_1, \text{Aut}(K_1)] & \xrightarrow{\beta} & [H_2, \text{Aut}(K_2)] \end{array}$$

where the maps $a_{(H_i, \text{Aut}(K_i))} : \frac{H_i}{L(H_i, \text{Aut}(K_i))} \times \text{Aut}(K_i) \rightarrow (H_i, \text{Aut}(K_i))$, for $i = 1, 2$, are given by

$$a_{(H_i, \text{Aut}(K_i))}(x_i L(H_i, \text{Aut}(K_i)), \alpha_i) = [x_i, \alpha_i].$$

Such a pair $(\psi \times \gamma, \beta)$ is said to be an autoisoclinism between the pairs of groups (H_1, K_1) and (H_2, K_2) . We conclude this paper with the following generalization of [3, Theorem 5.1] and [10, Lemma 2.5].

Theorem 5.1. *Let G_1 and G_2 be two finite groups with subgroups H_1, K_1 and H_2, K_2 respectively such that $H_1 \subseteq K_1$ and $H_2 \subseteq K_2$. If $(\psi \times \gamma, \beta)$ is an autoisoclinism between the pairs (H_1, K_1) and (H_2, K_2) then, for $g \in K_1$,*

$$\text{Pr}_g(H_1, \text{Aut}(K_1)) = \text{Pr}_{\beta(g)}(H_2, \text{Aut}(K_2)).$$

Proof. Let us consider the sets $\mathcal{S}_g = \{(x_1 L(H_1, \text{Aut}(K_1)), \alpha_1) \in \frac{H_1}{L(H_1, \text{Aut}(K_1))} \times \text{Aut}(K_1) : [x_1 L(H_1, \text{Aut}(K_1)), \alpha_1] = g\}$ and $\mathcal{T}_{\beta(g)} = \{(x_2 L(H_2, \text{Aut}(K_2)), \alpha_2) \in \frac{H_2}{L(H_2, \text{Aut}(K_2))} \times \text{Aut}(K_2) : [x_2 L(H_2, \text{Aut}(K_2)), \alpha_2] = \beta(g)\}$. Since (H_1, K_1) is autoisoclinic to (H_2, K_2) we have $|\mathcal{S}_g| = |\mathcal{T}_{\beta(g)}|$. Again, it is clear that

$$|\{(x_1, \alpha_1) \in H_1 \times \text{Aut}(K_1) : [x_1, \alpha_1] = g\}| = |L(H_1, \text{Aut}(K_1))| |\mathcal{S}_g| \quad (5.1)$$

and

$$|\{(x_2, \alpha_2) \in H_2 \times \text{Aut}(K_2) : [x_2, \alpha_2] = \beta(g)\}| = |L(H_2, \text{Aut}(K_2))| |\mathcal{T}_{\beta(g)}|. \quad (5.2)$$

Hence, the result follows from (1.1), (5.1) and (5.2). \square

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